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Fluid flow induced by a rapidly alternating or rotating magnetic field

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This paper studies the effect of alternating or rotating magnetic fields on containers of conducting fluid. The magnetic Reynolds number is assumed small. The frequency of alternation or rotation is rapid so the magnetic field is confined to a thin layer on the surface of the container. A boundary-layer analysis is used to find the rate of vorticity generation due to the Lorentz force. When the container is an infinitely long cylinder of uniform cross-section, alternating fields normal to the generators or fields rotating about an axis parallel to the generators generate vorticity at a constant rate. For containers of any other shape the rate of vorticity generation includes both constant and oscillatory terms. A perturbation analysis is used to study the flow induced in a slightly distorted circular cylinder by a rotating field. Complex flows develop in the viscous–magnetic boundary layer which may be unstable.

1. Introduction

The effect of a rotating magnetic field on a container of liquid metal has useful industrial applications, for example centrifuging to remove impurities or stirring of castings. Moffatt (1965) studied the effect of a rotating magnetic field on an infinitely long circular cylinder of conducting fluid. He assumed a low magnetic Reynolds number and found an exact expression for the magnetic field in terms of Bessel functions. Under the assumption that the field was rotating rapidly (compared with the magnetic diffusion time of the cylinder) the Bessel functions could be replaced by their large argument asymptotic expansions and a simpler expression derived for the magnetic field. It was found that the rate of vorticity generation by the Lorentz force $\nabla \times (\mathbf{j} \times \mathbf{B})$ was independent of time, and that this steady vorticity source produced a rigid-body rotation in the interior of the cylinder inside a viscous–magnetic boundary layer. Moffatt also made some conjectures about the effect of a rotating magnetic field on non-circular cylinders, which are examined in this paper.

Moffatt's work has been extended by other authors. Sneyd (1971) considered a circular cylinder in an alternating field. Nigam (1969) examined a spherical container of conducting fluid in a rotating magnetic field. His analysis is identical with Moffatt's (an exact Bessel-function solution for the field followed by asymptotic expansion) but there is a mistake in his paper which invalidates the conclusions. Dahlberg (1972) showed that the rate of vorticity generation in a circular cylinder was steady for all rotation rates of the magnetic field, and derived a general solution for the velocity field. The stability of this solution in the case of a slowly rotating field has been examined by Richardson (1974), who concluded that instability would occur for very

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small magnetic field strengths. A concise summary of these developments has been given by Moffatt (1978).

This paper examines the effect of alternating and rotating magnetic fields on containers of arbitrary shape. In § 2 a boundary-layer technique is used to find $\nabla \times (\mathbf{j} \times \mathbf{B})$ for an alternating field. It turns out that this is steady for infinitely long cylinders of arbitrary cross-section but not in general for finite three-dimensional containers, not even for the sphere. Section 3 deals with rotating fields and again $\nabla \times (\mathbf{j} \times \mathbf{B})$ is steady only in infinitely long cylinders. The solution of the Navier–Stokes equations to find the flow generated by such a steady vorticity source presents a difficult nonlinear problem. In Moffatt's circular cylinder the streamlines are circular so the inertial forces are balanced by a radial pressure gradient and a simple solution is possible. Section 4 analyses the flow produced by a rotating field in a slightly distorted circular cylinder, in which case it is possible to linearize the boundary-layer equations and obtain a perturbation solution.

2. Container of conducting fluid in a rapidly alternating magnetic field

The magnetic field

An arbitrarily shaped closed container of homogeneous fluid with electrical conductivity σ is placed in an alternating magnetic field $\mathrm{Re}\{\mathbf{B}_A e^{i\Omega t}\}$, $\mathbf{B}_A$ being a constant vector. We assume that the space outside the container is non-conducting and that the magnetic Reynolds number of the flow is small. If $\mathrm{Re}\{\mathbf{B} e^{i\Omega t}\}$ and $\mathrm{Re}\{\mathbf{B}^* e^{i\Omega t}\}$ are the magnetic fields inside and outside the fluid then

$$\nabla . \mathbf{B} = \nabla . \mathbf{B}^* = 0, \tag{2.1}$$

$$i\Omega \mathbf{B} = \lambda \nabla^2 \mathbf{B} \quad (\lambda = 1/\mu_0 \sigma), \tag{2.2}$$

$$\nabla \times \mathbf{B}^* = 0, \tag{2.3}$$

$$\mathbf{B} = \mathbf{B}^* \quad \text{on} \quad S, \tag{2.4}$$

$$\mathbf{B}^* \to \mathbf{B}_A \text{ at large distances from the container.} \tag{2.5}$$

S is the closed surface of the container.

Suppose that the magnetic field is alternating rapidly, i.e. that the period of oscillation is small compared with the magnetic diffusion time scale of the container. In the time $2\pi/\Omega$ required for one oscillation the field will diffuse into the fluid a distance of order $(\lambda/\Omega)^{\frac{1}{2}} = 2^{-\frac{1}{2}} \delta_B$, say, so the magnetic field in the fluid will be confined to a thin layer of width δ_B around S. If L is a typical diameter of the container the assumption of rapid oscillation is equivalent to assuming $\delta_B/L = \epsilon \ll 1$.

For the calculation of the magnetic field we shall use the following co-ordinate system. Let (p, q) be an orthogonal co-ordinate system on S such that the co-ordinate lines are lines of curvature. The equation of S can then be written in the parametric form $\mathbf{x} = \mathbf{x}_S(p, q)$. A point X is assigned co-ordinates (p, q, r) as follows. Let Y be the foot of the perpendicular from X to S. (Y is unique provided that X is sufficiently close to S.) Then (p, q) are the co-ordinates of Y on S and $r = YX$ (figure 1).

The position vector $\mathbf{x}$ of the point with co-ordinates (p, q, r) is given by

$$\mathbf{x} = \mathbf{x}_S(p, q) + r\hat{\mathbf{N}},$$

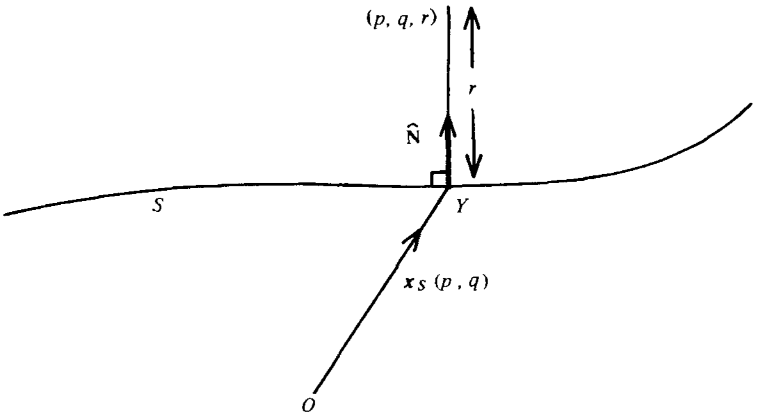

FIGURE 1

where $\hat{\mathbf{N}}(p,q)$ is the inward unit normal to S at (p, q). Now

$$\frac{\partial \mathbf{x}}{\partial p} = \frac{ds}{dp}\left(\frac{\partial \mathbf{x}_S}{\partial s} + r\frac{\partial \hat{\mathbf{N}}}{\partial s}\right) = \frac{ds}{dp}(1 - rK_p)\,\mathbf{e}_p, \tag{2.6}$$

where $s(p)$ measures arc length along the p co-ordinate line, K_p is a principal curvature and $\mathbf{e}_p = \partial\mathbf{x}_S/\partial s$ is a unit vector along the p co-ordinate line. Similarly, it can be shown that

$$\frac{\partial \mathbf{x}}{\partial q} = \frac{ds}{dq}(1 - rK_q)\mathbf{e}_q, \quad \frac{\partial \mathbf{x}}{\partial r} = \hat{\mathbf{N}}. \qquad (2.7), (2.8)$$

Equations (2.6)–(2.8) show that (p, q, r) is an orthogonal co-ordinate system ($\mathbf{e}_p$ and $\mathbf{e}_q$ being perpendicular since lines of curvature intersect at right angles) and that the scale factors are given by

$$h_p = \left|\frac{ds}{dp}(1 - rK_p)\right|, \quad h_q = \left|\frac{ds}{dq}(1 - rK_q)\right|, \quad h_r = 1.$$

We define a dimensionless orthogonal co-ordinate system (u, v, w) by setting

$$u = \frac{p}{L}\left(\frac{ds}{dp}\right)_{Y_0}, \quad v = \frac{q}{L}\left(\frac{ds}{dq}\right)_{Y_0}, \quad w = \frac{r}{\delta_B},$$

where $(ds/dp)_{Y_0}$ is ds/dp evaluated at some fixed point Y_0 on S. The scale factors are given by

$$h_u = h'_u L, \quad h_v = h'_v L, \quad h_w = \delta_B,$$

where h'_u and h'_v are the dimensionless functions

$$h'_u = \frac{ds/dp}{(ds/dp)_{Y_0}}(1 - \epsilon w K'_p), \quad h'_v = \frac{ds/dq}{(ds/dq)_{Y_0}}(1 - \epsilon w K'_q),$$

$K'_p = LK_p$ and $K'_q = LK_q$ being the dimensionless principal curvatures.

In terms of dimensionless variables (2.1) becomes

$$\frac{\partial}{\partial w}(h'_u h'_v B_w) + \epsilon\left[\frac{\partial}{\partial u}(h'_v B_u) + \frac{\partial}{\partial v}(h'_u B_v)\right] = 0. \tag{2.9}$$

If $\mathbf{B}$ is expanded as a power series in ϵ of the form

$$\mathbf{B} = \mathbf{B}_0 + \epsilon\mathbf{B}_1 + \epsilon^2\mathbf{B}_2 + \ldots,$$

the zeroth-order term of (2.9) is

$$\partial B_{0w}/\partial w = 0.$$

Since $\mathbf{B} \to 0$ as $w \to \infty$ (in the interior of the container) it follows that

$$B_{0w} = 0. \tag{2.10}$$

Equation (2.10) makes it possible in principle to determine the outer field $\mathbf{B}^*$ to zeroth order. Equations (2.1) and (2.3) show that one can write

$$\mathbf{B}_0^* = \nabla\phi \quad \text{with} \quad \nabla^2\phi = 0. \tag{2.11}$$

From (2.4) and (2.10) it follows that

$$\nabla\phi . \hat{\mathbf{N}} = 0 \quad \text{on} \quad S. \tag{2.12}$$

Equations (2.11) and (2.12) together with (2.5) determine $\mathbf{B}_0^*$ uniquely: in fact $\mathbf{B}_0^*$ is identical with the velocity field for irrotational flow of a uniform stream $\mathbf{B}_A$ past a rigid obstacle of the same shape as the container.

The zeroth-order part of (2.2) is

$$i\Omega(B_{0u}\,\mathbf{e}_u + B_{0v}\,\mathbf{e}_v) = \frac{\lambda}{\delta_B^2}\frac{\partial^2}{\partial w^2}(B_{0u}\,\mathbf{e}_u + B_{0v}\,\mathbf{e}_v).$$

Using the definition of δ_B, this becomes

$$(\partial^2/\partial w^2 - 2i)(B_{0u}\,\mathbf{e}_u + B_{0v}\,\mathbf{e}_v) = 0.$$

The boundary conditions are

$$B_{0u}\,\mathbf{e}_u + B_{0v}\,\mathbf{e}_v \to 0 \quad \text{as} \quad w \to \infty,$$

$$(B_{0u}\,\mathbf{e}_u + B_{0v}\,\mathbf{e}_v)_{w=0} = \mathbf{B}_S,$$

where $\mathbf{B}_S$ is the value of $\mathbf{B}^*$ on S determined from (2.11) and (2.12). The solution is

$$\mathbf{B}_0 = B_{0u}\,\mathbf{e}_u + B_{0v}\,\mathbf{e}_v = \mathbf{B}_S \exp[-w(1+i)], \tag{2.13}$$

so if $\mathbf{B}_I$ denotes the magnetic field in the interior of the fluid

$$\mathbf{B}_{I0} = \mathbf{B}_S\,e^{-w}\cos(\Omega t - w) = \mathbf{B}_S\,e^{-w}\cos\gamma, \quad \text{say}, \tag{2.14}$$

where $\gamma = \Omega t - w$.

For the calculation of the curl of the Lorentz force it is necessary to find B_{1w}. The first-order term of (2.9) is

$$h'_u h'_v \frac{\partial B_{1w}}{\partial w} + \frac{\partial}{\partial u}(h'_v B_{0u}) + \frac{\partial}{\partial v}(h'_u B_{0v}) = 0.$$

Substituting for B_{0u} and B_{0v} from (2.13) gives

$$\begin{aligned}\frac{\partial B_{1w}}{\partial w} &= -\frac{1}{h'_u h'_v}\left[\frac{\partial}{\partial u}(h'_v B_{Su}) + \frac{\partial}{\partial v}(h'_u B_{Sv})\right]\exp[-w(1+i)] \\ &= -\nabla'_S . \mathbf{B}_S \exp[-w(1+i)], \quad \text{say},\end{aligned}$$

where ∇'_S represents a non-dimensional surface gradient operator. $B_{1w} \to 0$ as $w \to \infty$ so

$$B_{1w} = \frac{\nabla'_S . \mathbf{B}_S}{1+i}\exp[-w(1+i)] = 2^{-\frac{1}{2}}\,\nabla'_S . \mathbf{B}_S \exp[-w(1+i) - \tfrac{1}{4}i\pi].$$

Thus

$$(\mathbf{B}_{I1})_w = 2^{-\frac{1}{2}}\,\nabla'_S . \mathbf{B}_S\,e^{-w}\cos(\gamma - \tfrac{1}{4}\pi). \tag{2.15}$$

The rate of vorticity generation

The zeroth-order component $\mathbf{j}_0$ of the current density $\mathbf{j} = \mu_0^{-1}\nabla\times\mathbf{B}$ is found from (2.14):

$$\mathbf{j}_0 = \frac{2^{\frac{1}{2}}}{\mu_0\,\delta_B}\,e^{-w}\,\mathbf{B}_S\times\mathbf{e}_w\cos\left(\gamma+\frac{\pi}{4}\right). \tag{2.16}$$

The important terms in the body force $\mathbf{F} = \mathbf{j}\times\mathbf{B}$ are F_{0w}, F_{1u} and F_{1v}. Each gives a contribution of the same order of magnitude to $\nabla\times\mathbf{F}$ since F_{0w} is differentiated with respect to u and v whereas F_{1u} and F_{1v} are differentiated with respect to w. Equations (2.14)–(2.16) give

$$F_{0w} = \frac{B_S^2}{2^{\frac{1}{2}}\mu_0\,\delta_B}\,e^{-2w}\left\{\cos\left(2\gamma+\frac{\pi}{4}\right)+2^{-\frac{1}{2}}\right\}, \tag{2.17}$$

$$F_{1u}\,\mathbf{e}_u + F_{1v}\,\mathbf{e}_v = \frac{-\mathbf{B}_S(\nabla_S'.\mathbf{B}_S)}{2\mu_0\,\delta_B}\,e^{-2w}\cos 2\gamma. \tag{2.18}$$

(Note that $j_{1w} = \mu_0^{-1}|\nabla\times\mathbf{B}_S|\,e^{-w}\cos\gamma = 0$ since there is no normal current flow across S, so $\nabla\times\mathbf{B}_S = 0$.) Evaluating $\nabla\times\mathbf{F}$ from (2.17) and (2.18) gives

$$\begin{aligned}(\nabla\times\mathbf{F})_0 = {} & \frac{1}{2^{\frac{1}{2}}\mu_0\,\delta_B L}\,\nabla_S'(B_S^2)\times\mathbf{e}_w\left\{\cos\left(2\gamma+\frac{\pi}{4}\right)+2^{-\frac{1}{2}}\right\}e^{-2w} \\ & +\frac{(\nabla_S'.\mathbf{B}_S)}{2\mu_0\,\delta_B}\,(\mathbf{B}_S\times\mathbf{e}_w)\,\frac{d}{dw}\,\{e^{-w}\cos 2\gamma\}.\end{aligned}$$

This can be written as the sum of a constant term and an oscillatory term as follows:

$$\begin{aligned}(\nabla\times\mathbf{F})_0 = {} & \frac{1}{2\mu_0\,\delta_B L}\,e^{-2w}\,\nabla_S'(B_S^2)\times\mathbf{e}_w + \frac{1}{2^{\frac{1}{2}}\mu_0\,\delta_B L}\,e^{-2w}\cos\left(2\gamma+\frac{\pi}{4}\right) \\ & \times\{\nabla_S'(B_S^2)\times\mathbf{e}_w - 2(\nabla_S'.\mathbf{B}_S)\,\mathbf{B}_S\times\mathbf{e}_w\}.\end{aligned} \tag{2.19}$$

Special cases

(i) *The infinitely long cylinder.* Suppose that the container is an infinitely long cylinder of uniform cross-section and that the applied magnetic field $\mathbf{B}_A$ is normal to its generators. If the z axis is chosen parallel to the generators and s measures arc length around the cross-section of the cylinder then we may take $u = s/L$ and $v = -z/L$. By symmetry we must have $\mathbf{B}_S = B_S(u)\,\mathbf{e}_u$, so

$$\nabla_S'(B_S^2) = 2B_S\,\frac{dB_S}{du}\,\mathbf{e}_u, \quad (\nabla_S'.\mathbf{B}_S)\,\mathbf{B}_S = B_S\,\frac{dB_S}{du}\,\mathbf{e}_u.$$

It follows that the oscillatory term of (2.19) vanishes and that

$$(\nabla\times\mathbf{F})_0 = \frac{1}{\mu_0\,\delta_B L}\,e^{-2w}\,B_S\,\frac{dB_S}{du}\,\mathbf{e}_z. \tag{2.20}$$

In the case of a circular cylinder of radius a, for example, we use polar co-ordinates (r, θ) with the x axis parallel to $\mathbf{B}_A$:

$$\mathbf{B}_S = -2B_A\sin\theta\,\mathbf{e}_\theta.$$

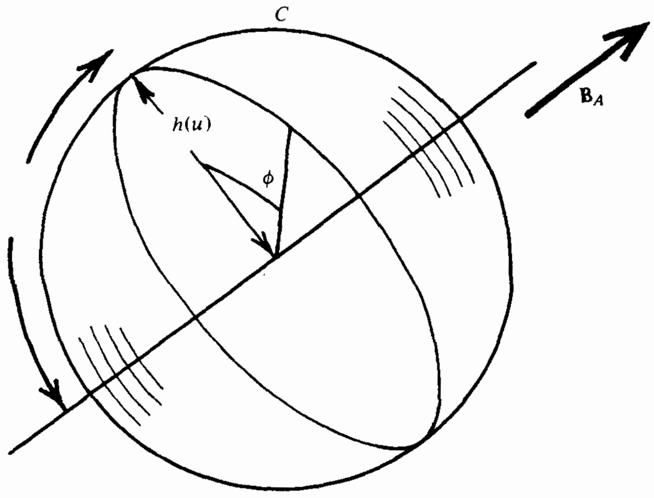

FIGURE 2

If we set $L = a$ then $u = \theta$ and

$$(\nabla \times \mathbf{F})_0 = \frac{2B_A^2}{\mu_0\, a\, \delta_B}\, e^{-2w} \sin 2\theta\, \mathbf{e}_z,$$

which is identical with equation (4.4) in Sneyd (1971), where $\beta = \delta_B^{-1}$.

(ii) *A body of revolution.* Suppose that S is obtained by rotating a curve C about an axis parallel to $\mathbf{B}_A$ (figure 2). If s measures arc length along C and ϕ is the azimuthal angle of rotation about the axis then one can set $u = s/L$ and $v = \phi$. Also, $h'_u = 1$ and $h'_v = h(u)/L$, where $h(u)$ is the perpendicular distance to the axis. By symmetry $\mathbf{B}_S = B_S(u)\, \mathbf{e}_u$, so

$$\nabla'_S(B_S^2) = 2B_S \frac{dB_S}{du}\, \mathbf{e}_u$$

and

$$(\nabla'_S \cdot \mathbf{B}_S)\mathbf{B}_S = \frac{1}{h(u)} \frac{d}{du}\, (h(u)\, B_S)\, B_S\, \mathbf{e}_u.$$

Substituting in (2.19) gives

$$(\nabla \times \mathbf{F})_0 = \frac{1}{\mu_0 \delta_B L}\, e^{-2w} B_S \frac{dB_S}{du}\, \mathbf{e}_u \times \mathbf{e}_w - \frac{2^{\frac{1}{2}} B_S^2\, h'(u)}{\mu_0 \delta_B L\, h(u)}\, e^{-2w} \cos\left(2\gamma + \frac{\pi}{4}\right) \mathbf{e}_u \times \mathbf{e}_w. \quad (2.21)$$

The rate of vorticity generation is not steady unless $h(u)$ is constant.

Consider, for example, a sphere of radius a. In terms of spherical polar co-ordinates (r, θ, ϕ) with the axis parallel to $\mathbf{B}_A$,

$$\mathbf{B}_S = -\tfrac{3}{2} B_A \sin\theta\, \mathbf{e}_\theta. \quad (2.22)$$

With $L = a$,

$$\nabla'_S(B_S^2) = \tfrac{9}{4} B_A^2 \sin 2\theta\, \mathbf{e}_\theta$$

and

$$(\nabla'_S \cdot \mathbf{B}_S)\mathbf{B}_S = \tfrac{9}{4} B_A^2 \sin 2\theta\, \mathbf{e}_\theta$$

so

$$(\nabla \times \mathbf{F})_0 = \frac{9B_A^2}{8\mu_0\, \delta_B\, a}\, e^{-2w} \sin 2\theta\, \mathbf{e}_\phi - \frac{9B_A^2}{4 \times 2^{\frac{1}{2}}\, \mu_0\, \delta_B\, a}\, e^{-2w} \sin 2\theta \cos\left(2\gamma + \frac{\pi}{4}\right) \mathbf{e}_\phi.$$

The exact solution for the sphere is

$$\mathbf{B} = \nabla \times \left(\frac{\psi}{r\sin\theta}\,\mathbf{e}_\phi\right),$$

where

$$\psi = \mathrm{Re}\left\{\frac{3B_A\,a}{2\alpha}\left(\frac{r}{a}\right)^{\frac{1}{2}}\frac{J_{\frac{3}{2}}(\alpha r)}{J_{\frac{1}{2}}(\alpha a)}\sin^2\theta\, e^{i\Omega t}\right\}, \quad \alpha = (1-i)/\delta_B.$$

When the Bessel functions are replaced by their large argument asymptotic expansions the expression for $\mathbf{B}$ is identical with that obtained by substituting (2.22) into (2.14) and (2.15).

Necessary condition for steady vorticity generation

The results just derived lead one to suspect that it is only in the case of an infinite cylinder that the rate of vorticity generation is steady, and this in fact turns out to be true.

It follows from (2.19) that $(\nabla\times\mathbf{F})_0$ is time independent only if

$$\nabla_S(B_S^2) = 2(\nabla_S.\mathbf{B}_S)\,\mathbf{B}_S. \tag{2.23}$$

Since there is no normal current flow across S

$$\nabla_S\times\mathbf{B}_S = 0, \tag{2.24}$$

so that

$$\mathbf{B}_S = \nabla_S\,\chi(u,v).$$

One can construct an orthogonal co-ordinate system (ξ,η) on S by letting the ξ co-ordinate curves be field lines of $\mathbf{B}_S$ and the η co-ordinate curves level cuives of χ. Then $\mathbf{B}_S = B_S\,\mathbf{e}_\xi$ and (2.23) shows that $\nabla_S B_S$ is parallel to $\mathbf{B}_S$, which implies that B_S is a function of ξ only, $A(\xi)$, say. Thus

$$\mathbf{B}_S = A(\xi)\,\mathbf{e}_\xi.$$

Substitution of this expression in (2.23) and (2.24) yields respectively

$$\partial h_\eta/\partial\xi = 0, \quad \partial h_\xi/\partial\eta = 0. \tag{2.25}$$

The Gaussian curvature K of S is given by the formula

$$K = -\frac{1}{h_\eta h_\xi}\left[\frac{\partial}{\partial\xi}\left(\frac{1}{h_\xi}\frac{\partial h_\eta}{\partial\xi}\right)+\frac{\partial}{\partial\eta}\left(\frac{1}{h_\eta}\frac{\partial h_\xi}{\partial\eta}\right)\right]$$

(see, for example, Spivak 1970, p. 322) and it follows that

$$K\equiv 0 \quad \text{on} \quad S.$$

It is a well-known result in differential geometry (see, for example, Spivak 1970, theorem 9, p. 363) that the only 'flat' surfaces, i.e. surfaces with Gaussian curvature everywhere zero, are infinite cylinders, so steady vorticity generation is possible only in this case. For the 'flat' surfaces the magnetic field and electric current are perpendicular and 90 ° out of phase, giving a steady $\nabla\times(\mathbf{j}\times\mathbf{B})$. For any finite closed surface, the extra curvature which it must necessarily possess makes this configuration impossible.

Even in the case of a cylinder the vorticity generation rate is constant only if the direction of the applied field is parallel or normal to the generators. For suppose that $\mathbf{B}_A$ makes an angle α with the generators. In the notation of special case (i)

$$\mathbf{B}_S = \cos\alpha\, B_S(u)\mathbf{e}_u + B_A\sin\alpha\,\mathbf{e}_z$$

so
$$B_S^2 = \cos^2\alpha\, B_S^2(u) + B_A^2 \sin^2\alpha.$$
Evidently (2.23) can be satisfied only if $\alpha = 0$ or $\frac{1}{2}\pi$.

3. Container of conducting fluid in a rapidly rotating magnetic field

A rotating magnetic field can be represented by
$$\mathbf{B} = \mathbf{B}_A \cos \Omega t + \mathbf{B}_P \sin \Omega t,$$
where $\mathbf{B}_A$ and $\mathbf{B}_P$ are perpendicular vectors of equal magnitude. The effect of this field on a container of conducting fluid can be deduced from the results of the previous section. The differential equations and boundary conditions determining $\mathbf{B}$ are linear so (2.14) and (2.15) give
$$\mathbf{B}_{I0} = \mathbf{B}_S e^{-w} \cos\gamma + \mathbf{B}_T e^{-w} \sin\gamma, \tag{3.1}$$
$$(\mathbf{B}_{I1})_w = (\nabla'_S.\mathbf{B}_S)\, 2^{-\frac{1}{2}} e^{-w} \cos\left(\gamma - \frac{\pi}{4}\right) + (\nabla'_S.\mathbf{B}_T)\, 2^{-\frac{1}{2}} e^{-w} \sin\left(\gamma - \frac{\pi}{4}\right). \tag{3.2}$$
$\mathbf{B}_T(u, v)$ is the counterpart of $\mathbf{B}_S$ produced by $\mathbf{B}_P$. ($\mathbf{B}_T$ is the surface value of a vector field which is irrotational and solenoidal outside S, has zero normal component on S and tends to $\mathbf{B}_P$ at infinity.) Equation (2.16) gives the electric current density:
$$\mathbf{j}_0 = \frac{2^{\frac{1}{2}}}{\mu_0 \delta_B} e^{-w} \left[\mathbf{B}_S \cos\left(\gamma + \frac{\pi}{4}\right) + \mathbf{B}_T \sin\left(\gamma + \frac{\pi}{4}\right)\right] \times \mathbf{e}_w.$$
The Lorentz force $\mathbf{j} \times \mathbf{B}$ is quadratic in $\mathbf{B}$ so one can write
$$\mathbf{F} = \mathbf{F}_S + \mathbf{F}_T + \mathbf{F}_C,$$
where $\mathbf{F}_S$ and $\mathbf{F}_T$ are the terms involving only $\mathbf{B}_S$ and $\mathbf{B}_T$, and $\mathbf{F}_C$ represents the cross-product terms:
$$\mathbf{F}_C = \frac{2^{\frac{1}{2}}}{2\mu_0 \delta_B} e^{-2w} \sin\left(2\gamma + \frac{\pi}{4}\right) (\mathbf{B}_S.\mathbf{B}_T)\,\mathbf{e}_w$$
$$- \frac{\epsilon}{2\mu_0 \delta_B} e^{-2w} [(\sin 2\gamma - 1)(\nabla'_S.\mathbf{B}_T)\mathbf{B}_S + (\sin 2\gamma + 1)(\nabla'_S.\mathbf{B}_S)\mathbf{B}_T]$$
and
$$\nabla \times \mathbf{F}_C = \frac{2^{\frac{1}{2}} e^{-2w}}{\mu_0 \delta_B L} \sin\left(2\gamma + \frac{\pi}{4}\right) [\nabla'_S(\mathbf{B}_S.\mathbf{B}_T) - (\nabla'_S.\mathbf{B}_T)\mathbf{B}_S - (\nabla'_S.\mathbf{B}_S)\mathbf{B}_T]$$
$$\times\, \mathbf{e}_w + \frac{e^{-2w}}{\mu_0 \delta_B L} [(\nabla'_S.\mathbf{B}_T)\mathbf{B}_S - (\nabla'_S.\mathbf{B}_S)\mathbf{B}_T] \times \mathbf{e}_w. \tag{3.3}$$
$\nabla \times \mathbf{F}_S$ is given by the right-hand side of (2.19) and $\nabla \times \mathbf{F}_T$ is the same expression with $\mathbf{B}_S$ replaced by $\mathbf{B}_T$ and the sign of the oscillatory term reversed.

The sphere

We use spherical polar co-ordinates with the z axis parallel to the axis of rotation. If the applied rotating field is
$$B_A \mathbf{i} \cos \Omega t + B_A \mathbf{j} \sin \Omega t$$
then
$$\mathbf{B}_S = \tfrac{3}{2} B_A [\cos\theta \cos\phi\, \mathbf{e}_\theta - \sin\phi\, \mathbf{e}_\phi],$$
$$\mathbf{B}_T = \tfrac{3}{2} B_A [\cos\theta \sin\phi\, \mathbf{e}_\theta + \cos\phi\, \mathbf{e}_\phi].$$

Substituting these expressions in (2.19) and (3.3) gives

$$(\nabla\times\mathbf{F})_0 = -\frac{9B_A^2\,e^{-2w}}{8\mu_0\,\delta_B\,a}\,[\sin 2\theta\,\mathbf{e}_\phi + 4\sin\theta\,\mathbf{e}_\theta]$$

$$+\frac{9\times 2^{\frac{1}{2}}B_A^2}{8\mu_0\,\delta_B\,a}\,e^{-2w}\left[\sin 2\theta\cos\left(2\gamma+\frac{\pi}{4}-2\phi\right)\mathbf{e}_\phi - 2\sin\theta\sin\left(2\gamma+\frac{\pi}{4}-2\phi\right)\mathbf{e}_\theta\right].$$

This expression is different from that given by Nigam (1969) in his equation (3.1). The reason for the difference is that Nigam has incorrect expressions for the magnetic field in his equations (2.19). A factor of $\frac{1}{2}$ is missing from (2.19*b*) and (2.19*c*). (It is easily verified that Nigam's expression for $\mathbf{B}$ is incorrect since it does not satisfy $\nabla.\mathbf{B} = 0$.) The omission of this factor of $\frac{1}{2}$ leads to a spurious cancellation of the oscillatory terms in Nigam's expression for $\nabla\times\mathbf{F}$.

The infinite cylinder

For an infinitely long cylinder of uniform cross-section with the magnetic field rotating about an axis parallel to its generators we use the co-ordinate system $u = s/L$, $v = -z/L$, w, as in § 2. It follows from symmetry that

$$\mathbf{B}_S = B_S(u)\,\mathbf{e}_u,\quad \mathbf{B}_T = B_T(u)\,\mathbf{e}_u$$

so

$$\nabla'_S(\mathbf{B}_S.\mathbf{B}_T) - (\nabla'_S.\mathbf{B}_S)\,\mathbf{B}_T - (\nabla'_S.\mathbf{B}_T)\,\mathbf{B}_S = \left[\frac{d}{du}(B_S B_T) - \frac{dB_S}{du}B_T - \frac{dB_T}{du}B_S\right]\mathbf{e}_u = 0.$$

Equations (3.3) and (2.20) give

$$(\nabla\times\mathbf{F})_0 = \frac{e^{-2w}}{\mu_0\,\delta_B\,L}\left[\frac{d}{du}\left(\tfrac{1}{2}B_S^2+\tfrac{1}{2}B_T^2\right) + B_S\frac{dB_T}{du} - B_T\frac{dB_S}{du}\right]\mathbf{e}_z. \tag{3.4}$$

Moffatt (1965) conjectured that for a cylinder of arbitrary cross-section $\nabla\times\mathbf{F}$ might in general include an oscillatory term, but (3.4) shows that it is always time independent. Moffatt also conjectured that the magnitude of $\nabla\times\mathbf{F}$ would be proportional to the curvature of the cylinder surface but (3.4) illustrates a more obscure dependence on u.

For a circular cylinder of radius a,

$$\mathbf{B}_S = -2B_A\sin\theta\,\mathbf{e}_\theta,\quad \mathbf{B}_T = 2B_A\cos\theta\,\mathbf{e}_\theta.$$

Substitution in (3.4) gives

$$(\nabla\times\mathbf{F})_0 = \frac{4B_A^2\,e^{-2w}}{\mu_0\,\delta_B\,a}\,\mathbf{e}_z.$$

This is equivalent to equation (2.15) of Moffatt (1965), where $k = 1/\delta_B$ and the factor $1/4\pi$ appears because of different electromagnetic units.

Necessary condition for steady vorticity generation

The curl of the Lorentz force is given by

$$\nabla\times\mathbf{F} = \nabla\times\mathbf{F}_S + \nabla\times\mathbf{F}_T + \nabla\times\mathbf{F}_C.$$

The time-dependent parts of $\nabla\times\mathbf{F}_S$ and $\nabla\times\mathbf{F}_T$ are proportional to $\cos(2\gamma+\frac{1}{4}\pi)$ and that of $\nabla\times\mathbf{F}_C$ is proportional to $\sin(2\gamma+\frac{1}{4}\pi)$ so cancellation between these is impossible. It follows that $\nabla \times \mathbf{F}$ is steady only if both $\nabla \times \mathbf{F}_S$ and $\nabla \times \mathbf{F}_T$ are. As shown in § 2, this implies that the Gaussian curvature of S is everywhere zero, and that steady vorticity generation occurs only in the case of the cylinder.

The slightly distorted circular cylinder

Consider the cylinder

$$r = a[1+\delta f(\theta)] = r_S, \quad \text{say} \tag{3.5}$$

(cylindrical polar co-ordinates), where

$$\delta \ll 1$$

and

$$f(\theta) = \sum_{n=2}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) = \sum_{n=-\infty}^{\infty} \alpha_n e^{in\theta} \quad (\alpha_0 = \alpha_{\pm 1} = 0).$$

(The $n = 1$ terms are neglected in $f(\theta)$ since they correspond simply to a displacement of the centre, and in fact provide no contribution to the rate of vorticity generation.)

Define ϕ to be the solution of the problem

$$\nabla^2\phi = 0, \quad (\nabla\phi.\mathbf{n})_{r=r_S} = 0,$$
$$\phi \to B_A re^{i\theta} \quad \text{as} \quad r \to \infty,$$

where $\mathbf{n}$ is a unit normal to the cylinder surface. Then

$$(\nabla\phi.\mathbf{e}_u)_{r=r_S} = B_S + iB_T = S(u), \quad \text{say.}$$

It follows that

$$\frac{1}{2}\frac{d}{du}(B_S^2+B_T^2) + \frac{dB_T}{du}B_S - \frac{dB_S}{du}B_T = \text{Re}\left[\bar{S}\frac{dS}{du}\right] + \text{Im}\left[\bar{S}\frac{dS}{du}\right]. \tag{3.6}$$

We expand ϕ and S as power series in δ of the form

$$\phi = \phi_0 + \delta\phi_1 + \delta^2\phi_2 + \ldots, \quad S = S_0 + \delta S_1 + \delta^2 S_2 + \ldots.$$

Now

$$\phi_0 = B_A(r + a/r)e^{i\theta},$$

so ϕ_1 satisfies the following equations:

$$\nabla^2\phi_1 = 0,$$
$$\phi_1 \to 0 \quad \text{as} \quad r \to \infty,$$
$$\left(\frac{\partial\phi_1}{\partial r}\right)_{r=a} = 2B_A i \frac{d}{d\theta}[f(\theta)e^{i\theta}].$$

The solution is

$$\phi_1 = 2B_A \sum_{n=-\infty}^{\infty} \alpha_n \frac{a^{|n+1|+1}}{r^{|n+1|}} \operatorname{sgn}(n+1)\exp[i(n+1)\theta], \tag{3.7}$$

and

$$S_0 = 2B_A ie^{i\theta}, \quad S_1 = -2B_A if(\theta)e^{i\theta} + \frac{1}{a}\left(\frac{\partial\phi_1}{\partial\theta}\right)_{r=a}.$$

Substituting into (3.4) using (3.6) and (3.7) yields

$$|(\nabla \times \mathbf{F})_0| = \frac{4B_A^2 e^{-2\iota\iota}}{\mu_0 a\delta_B}[1 + \delta F(\theta) + O(\delta^2)], \tag{3.8}$$

where

$$F(\theta) = \sum_{n=2}^{\infty} (n-1)\{a_n(3\cos n\theta - n\sin n\theta) + b_n(3\sin n\theta + n\cos n\theta)\}. \tag{3.9}$$

$r = a(1+\delta\cos 2\theta)$ approximately represents an ellipse of small eccentricity. According to (3.8) the rate of vorticity generation in a cylinder of this cross-section is

$$\frac{4B_A^2\,e^{-2w}}{\mu_0\,\delta_B\,a}[1+\delta(3\cos 2\theta - 2\sin 2\theta)] = \frac{4B_A^2\,e^{-2w}}{\mu_0\,\delta_B\,a}[1+\delta 13^{\frac{1}{2}}\cos(2\theta+\alpha)],$$

where $\alpha = \tan^{-1}\frac{2}{3} = 33{\cdot}7^\circ$. This illustrates the falseness of Moffatt's conjecture that the rate of vorticity generation is proportional to the curvature. $|\nabla\times\mathbf{F}|$ attains its maximum when $\theta = -16{\cdot}8$ *or* $163{\cdot}2^\circ$.

The magnetic field strength on the cylinder surface is

$$B_S\cos\Omega t + B_T\sin\Omega t = A\cos(\Omega t - \alpha), \quad \text{say}.$$

The expression (3.4) for the rate of vorticity generation can be re-written in terms of the magnitude A and the phase α as follows:

$$(\nabla\times\mathbf{F})_0 = \frac{e^{-2w}}{\mu_0\,\delta_B\,L}A\left(\frac{dA}{du}+A\frac{d\alpha}{du}\right)\mathbf{e}_z. \tag{3.10}$$

It can be seen from (3.10) that $|\nabla\times\mathbf{F}|$ depends upon A, dA/du and $d\alpha/du$. In the case of a circular cylinder $A = 2B_A$ and $\alpha = u$ so $|\nabla\times\mathbf{F}|$ is uniform. For an elliptic cylinder of small eccentricity

$$A = 2B_A(1+\delta\cos 2\theta)+O(\delta^2),$$
$$d\alpha/du = 1+2\delta\cos 2\theta + O(\delta^2).$$

In this case all three terms A, dA/du and $d\alpha/du$ vary with u. A and $d\alpha/du$ attain a maximum at $\theta = 0$ and dA/du at $\theta = \frac{1}{4}\pi$. Their combined effect is maximum at some intermediate point, which explains the asymmetry of the point of maximum vorticity generation.

4. Flow induced in a slightly distorted circular cylinder by a rotating magnetic field

General remarks

Moffatt's solution for the flow in a circular cylinder due to a rotating field includes no inertial effects since the streamlines are circular, so it is of interest to study non-circular cylinders. To obtain a tractable problem we suppose that the cylinder is only slightly distorted from a circular shape, in fact that its surface is described by (3.5). This perturbation analysis is similar to that by Wood (1956) of boundary layers with closed streamlines.

The rate of vorticity generation is given by (3.8) and our aim here is to calculate the steady flow produced. The flow can be divided into two regions (figure 3): an inner region I where no body forces act and a boundary layer II where magnetic and viscous forces are important.

We shall suppose that the Reynolds number $R = Ua/\nu$ for the flow in the inner region is large. (For mercury in a container of radius 10^{-1} m, $R \gg 1$ if $U \gg 10^{-6}$ m/s, which is the only situation of physical interest.) As suggested by Moffatt (1965), the

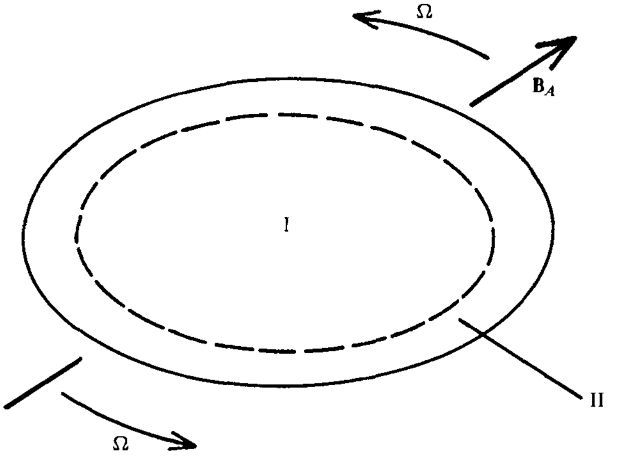

FIGURE 3

flow there can then be found using a result of Batchelor (1959) which shows that the vorticity must be uniform if one assumes the streamlines in the interior are closed. Thus if ψ_I is the stream function in region I we have

$$\left.\begin{aligned} \nabla^2 \psi_I &= \omega_I \quad \text{(a constant)}, \\ \psi_I &= 0 \quad \text{at} \quad r = a[1+\delta f(\theta)]. \end{aligned}\right\} \tag{4.1}$$

To first order in δ the solution of (4.1) is

$$\psi_I = \tfrac{1}{4}\omega_I(r^2-a^2) - \tfrac{1}{2}\delta\omega_I a^2 \sum_{n=2}^{\infty}\left[a_n\left(\frac{r}{a}\right)^n \cos n\theta + b_n\left(\frac{r}{a}\right)^n \sin n\theta\right], \tag{4.2}$$

the constant ω_I being determined by the boundary layer.

Boundary-layer equations

Region II includes two boundary layers: a magnetic layer of thickness δ_B and a viscous layer of thickness δ_V, say, where

$$\delta_V^2 = \nu a/U,$$

in which
$$U = B_A^2\,\delta_B^2/(2\rho\mu_0\,\nu a) \tag{4.3}$$

is a typical flow speed in the boundary layer. We define δ_L, the overall boundary-layer thickness, to be the maximum of δ_B and δ_V and set $k = \delta_L/\delta_B$. δ_V and δ_B are determined by the zeroth-order Moffatt solution and are independent of δ.

It can be seen, for example by integrating the Navier–Stokes equation around a closed streamline passing through the boundary layer, that the rotational Lorentz force can be balanced only by viscous forces, so in the magnetic boundary layer viscous and magnetic forces must be of the same order of magnitude. If $\delta_V > \delta_B$ the inertial forces there are smaller than this common magnitude and if $\delta_B > \delta_V$ they are larger. In theory the ratio δ_B/δ_V can be varied arbitrarily (for example by changing the magnetic field intensity and rotation speed) but in practice one would expect the regime $\delta_B > \delta_V$ to be unstable. The Taylor number T for the magnetic boundary layer is given by equation (3.1) of Moffatt (1965), and algebraic rearrangement shows that

$$T = \epsilon^{-1}\delta_B^2/\delta_V^2.$$

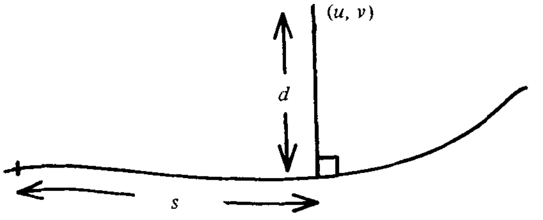

FIGURE 4

Moffatt argues that instability of the boundary layer would be expected for $T > \alpha \times 10^3$ (α being a number of order of magnitude 1) so it follows that δ_B cannot be much larger than δ_T. Further evidence of the instability of this regime is given later in this section.

We define $\epsilon^* = \delta_L/a$ and shall suppose that $\delta \gg \epsilon^*$, i.e. that in the change from a circular to a slightly non-circular cross-section the cylinder walls have been perturbed through a distance which is large compared with the boundary-layer thickness δ_L.

Essentially the same co-ordinate system as in § 2 will be used. s is arc length around the surface of the cylinder and d is perpendicular distance from the surface (figure 4). If one sets $u = s/a$ and $v = d/\delta_L = w/k$ then (u, v) is an orthogonal co-ordinate system with scale factors

$$h_u = (1-\epsilon^* \kappa' v), \quad h_v = \delta_L,$$

κ' being a dimensionless curvature.

If ψ' is the stream function for the boundary-layer flow a dimensionless stream function is defined by $\psi = \psi'/(U\delta_L)$. Then ψ, the fluid velocity and the fluid vorticity are expanded as power series in δ of the form

$$\begin{aligned} \psi &= \psi_0 + \delta\psi_1 + \delta^2\psi_2 + \dots, \\ \mathbf{u} &= \mathbf{u}_0 + \delta\mathbf{u}_1 + \delta^2\mathbf{u}_2 + \dots, \\ \boldsymbol{\omega} &= \boldsymbol{\omega}_0 + \delta\boldsymbol{\omega}_1 + \delta^2\boldsymbol{\omega}_2 + \dots . \end{aligned}$$

To first order in δ the curl of the steady Navier–Stokes equation is

$$\begin{aligned} &\nabla \times (\boldsymbol{\omega}_0 \times \mathbf{u}_0) + \delta\nabla \times (\boldsymbol{\omega}_1 \times \mathbf{u}_0) + \delta\nabla \times (\boldsymbol{\omega}_0 \times \mathbf{u}_1) \\ &\qquad = (4B_0^2 e^{-2kv}/a\mu_0\delta_B)[1+\delta F(\theta)]\,\mathbf{e}_z - \nu\nabla \times \nabla \times (\boldsymbol{\omega}_0 + \delta\boldsymbol{\omega}_1). \end{aligned}$$

According to Moffatt (1965)

$$\mathbf{u}_0 = U(1-e^{-2kv})\,\mathbf{e}_u. \tag{4.4}$$

Substituting (4.4) into the vorticity equation gives

$$\begin{aligned} &\frac{\delta U^2}{\delta_B a}\left[(e^{-2kv}-1)\frac{\partial^3\psi_1}{\partial v^2\,\partial u} - 4k^2 e^{-2kv}\frac{\partial\psi_1}{\partial u}\right] \\ &\qquad = \frac{4B_A^2 e^{-2kv}}{\rho\mu_0\delta_B a}[1+\delta F(\theta)] - \frac{8k^3 U\nu}{\delta_B^3}e^{-2kv} - \frac{\delta U\nu}{\delta_B^3}\frac{\partial^4\psi_1}{\partial v^4}. \end{aligned} \tag{4.5}$$

Equation (4.3) ensures that the zeroth-order part of (4.5) is satisfied. Since $\theta = u + O(\delta)$ the first-order terms of (4.5) give

$$R_L\left[(e^{-2kv}-1)\frac{\partial^3\psi_1}{\partial v^2\,\partial\theta} - 4k^2 e^{-2kv}\frac{\partial\psi_1}{\partial\theta}\right] = 8k^3 e^{-2kv}F(\theta) - \frac{\partial^4\psi_1}{\partial v^4}, \tag{4.6}$$

where $R_L = U\delta_L^2/a\nu$ is the boundary-layer Reynolds number. If one writes

$$\psi_1 = \sum_{n=2}^{\infty} \mathrm{Re}\,[f_n(v)\,e^{in\theta}]$$

(4.6) becomes a sequence of ordinary differential equations:

$$niR_L[(e^{-2kv}-1)f_n''-4k^2e^{-2kv}f_n] = 8k^3A_n e^{-2kv}-f_n^{(\mathrm{iv})} \quad (n=2,3,\ldots), \tag{4.7}$$

where

$$A_n = (n-1)[a_n(3+in)+b_n(n-3i)].$$

Matching the core region with the boundary layer

In (4.2), ω_I will be a function of δ and can be expanded in the form

$$\omega_I = \omega_{I0}+\delta\omega_{I1}+\delta^2\omega_{I2}+\ldots.$$

The flow speed q_I at the outer boundary of region I can be calculated from (4.2):

$$q_I = \tfrac{1}{2}a\omega_I-\tfrac{1}{2}\omega_I a\delta\sum_{n=2}^{\infty}(na_n\cos n\theta+nb_n\sin n\theta),$$

or

$$q_I = \tfrac{1}{2}a\omega_{I0}+\tfrac{1}{2}a\delta\left[\omega_{I1}-\omega_{I0}\sum_{n=2}^{\infty}(na_n\cos n\theta+nb_n\sin n\theta)\right]+O(\delta^2). \tag{4.8}$$

q_I must be equal to the flow speed at the inner edge of the boundary layer so

$$q_I = \mathrm{Re}\left\{U\left[1+\delta\sum_{n=2}^{\infty}f_n'(\infty)e^{in\theta}\right]\right\}. \tag{4.9}$$

Comparison of (4.8) and (4 9) gives

$$\omega_{I0} = 2U/a = B_A^2\delta_B^2/(\rho\mu_0a^2), \quad \omega_{I1}=0, \tag{4.10a, b}$$

$$f_n'(\infty) = -n(a_n-ib_n) = F_n, \quad \text{say.} \tag{4.10c}$$

The first of these equations is equivalent to equation (2.20) of Moffatt (1965). The second is analogous to equation (12) of Wood (1956), which shows essentially that the first-order perturbation to the core flow is zero.

Determination of $f_n(v)$

The fluid velocity must vanish on the cylinder wall so

$$f_n(0) = f_n'(0) = 0. \tag{4.11}$$

Equation (4.7) can be integrated once with respect to v, giving

$$niR_L[(e^{-2kv}-1)f_n'+2ke^{-2kv}f_n+F_n] = -4k^2A_ne^{-2kv}-f_n''', \tag{4.12}$$

the arbitrary constant of integration being fixed by (4.10c). Equations (4.10c)–(4.12) now determine $f_n(v)$ uniquely.

It is possible to find approximate solutions of (4.12) when $\delta_V \gg \delta_B$ or $\delta_B \gg \delta_V$.

Case 1: $\delta_V \gg \delta_B$. Since $\delta_L = \delta_V$, $R_L = 1$ and $k = \delta_L/\delta_B \gg 1$. The boundary layer divides into an inner region of thickness k^{-1} at $v=0$ (the magnetic boundary layer) and an outer region. If one writes $\epsilon_k = k^{-1}$, (4.12) becomes

$$ni\left[[\exp(-2v/\epsilon_k)-1]f_n'+\frac{2}{\epsilon_k}\exp(-2v/\epsilon_k)f_n+F_n\right] = \frac{-4A_n}{\epsilon_k^2}\exp(-2v/\epsilon_k)-f_n'''. \tag{4.13}$$

An approximate solution of (4.13) can be constructed by the method of matched asymptotic expansions and leads to the following uniformly valid expansion for $f'_n(v)$:

$$f'_n(v) = -A_n e^{-2kv} + (A_n - F_n) e^{-\beta v} + F_n \quad [\beta = (1+i)(\tfrac{1}{2}n)^{\frac{1}{2}}]. \tag{4.14}$$

For the elliptic cylinder $r = a(1+\delta\cos 2\theta)$,

$$f'_2(v) = -(3+2i)e^{-2kv} + (4+2i)e^{-(1+i)v} - 1.$$

Graphs of the real and imaginary parts of $f'_2(v)$ are drawn in figure 5. These represent respectively the boundary-layer velocity profiles at $\theta = 0$ and $\theta = \frac{1}{4}\pi$.

In this regime inertial forces are unimportant in the magnetic boundary layer, where the first-order variation in the flow is due to the first-order Lorentz-force term. In the outer viscous boundary layer the important influences on the first-order flow are viscosity, inertia and the perturbation in the core flow. The effect is an oscillatory velocity profile, the necessity for which was pointed out by Wood (1956, p. 82).

Case 2: $\delta_B \gg \delta_V$. $\delta_L = \delta_B$ so $k = 1$ and $R_L \gg 1$. Equation (4.12) now becomes

$$niR_L[(e^{-2v}-1)f'_n + 2e^{-2v}f_n + F_n] = -4A_n e^{-2v} - f'''_n. \tag{4.15}$$

Finding an asymptotic expansion for $f_n(v)$ in terms of the small parameter R_L^{-1} is a singular perturbation problem. There is a region of thickness $O(R_L^{-\frac{1}{3}})$ near $v = 0$ in which $R_L^{-1}f'''_n$ will be of the same order of magnitude as f_n. We set $\epsilon_R = R_L^{-\frac{1}{3}}$ and (4.15) becomes

$$in[(e^{-2v}-1)f'_n + 2e^{-2v}f_n + F_n] = -4\epsilon_R^3 A_n e^{-2v} - \epsilon_R^3 f'''_n. \tag{4.16}$$

Again an approximate solution of (4.16) can be calculated by the method of matched asymptotic expansions, resulting in the following uniformly valid expansion for $f'_n(v)$:

$$\begin{aligned} f'_n(v) = {} & \frac{\alpha F_n}{2\epsilon_R C_2}\left[\tfrac{1}{3}e^{-2v} - \mathrm{Ai}_1\left(\frac{\alpha v}{\epsilon_R}\right)\right] + F_n(1-e^{-2v}) \\ & \times\left[-G_0\alpha + \log\left(\frac{2\epsilon_R}{\alpha}\right) - \log(e^{2v}-1)\right] + F_n\log\left(\frac{e^{2v}-1}{2v}\right) + F_n\alpha g'\left(\frac{\alpha v}{\epsilon_R}\right), \end{aligned} \tag{4.17}$$

where

$$\mathrm{Ai}_1(x) = \int_x^\infty \mathrm{Ai}(t)\,dt, \quad \mathrm{Ai}_2(x) = \int_x^\infty \mathrm{Ai}_1(t)\,dt,$$

$\mathrm{Ai}(x)$ being the Airy function of the first kind (defined, for example, on p. 446 of Abramowitz & Stegun 1965) and $g(x)$ the solution of the problem

$$g''' - xg' + g = \frac{-x}{\alpha} + \frac{1}{2\alpha C_2}[2x\,\mathrm{Ai}_2(x) + x^2\,\mathrm{Ai}_1(x)],$$

$$g(0) = g'(0) = 0.$$

$$\alpha = (2n)^{\frac{1}{3}}e^{\frac{1}{6}i\pi}, \quad C_2 = 3^{-\frac{1}{3}}/\Gamma(\tfrac{1}{3}), \quad G_0 = 0{\cdot}1175 + 0{\cdot}0134i.$$

Figure 6 shows graphs of the boundary-layer velocity profiles calculated from (4.17), which indicate the presence of a jet increasing in intensity as $\epsilon_R \to 0$. It was pointed out earlier in this section that one might expect this regime to be unstable. The magnetic-boundary-layer Reynolds number is large and once the container departs from a pure circle inertia forces dominate both the viscous and the Lorentz forces (which are of the same order of magnitude). The most important dynamic effect in

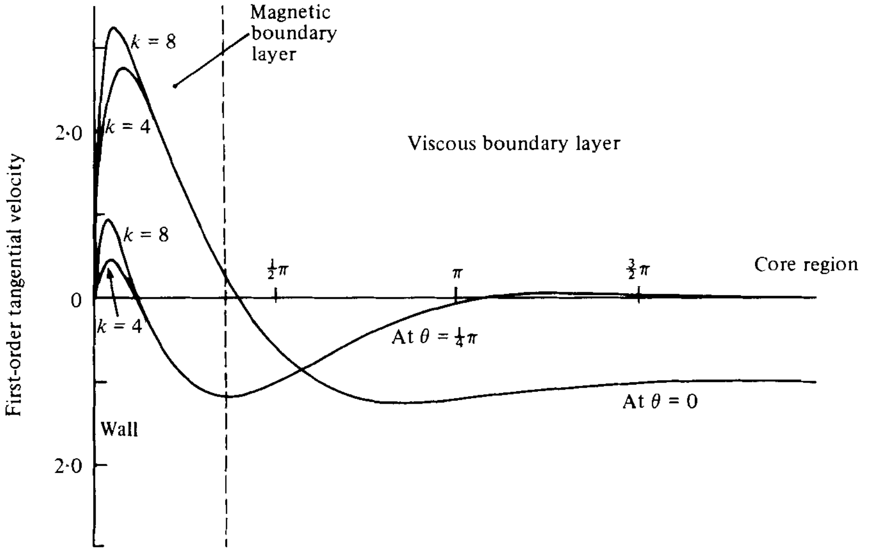

FIGURE 5. Boundary-layer velocity profiles for case (i) ($\delta_V \gg \delta_B$).

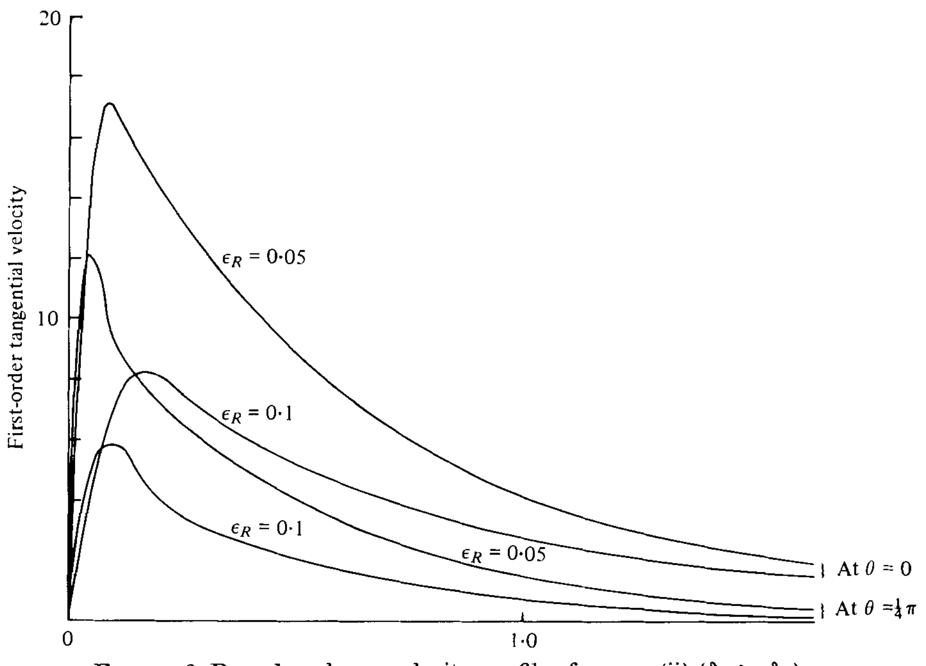

FIGURE 6. Boundary-layer velocity profiles for case (ii) ($\delta_B \gg \delta_V$).

this boundary layer is the distortion of the zeroth-order (in δ) Moffatt profile. Thus (4.17) shows that the first-order profiles are independent of the first-order Lorentz-force term.

5. Discussion

The precise conditions for the validity of the low magnetic Reynolds number assumption ($R_m \ll 1$) can now be found. $R_m = UL/\lambda$ and the typical flow speed U is given by (4.3) so

$$R_m = \frac{\sigma B_A^2 L^2}{\rho\nu}\left(\frac{\lambda}{\Omega L^2}\right) = 2M^2\epsilon^2,$$

where $M = B_A L(\sigma/\rho\nu)^{\frac{1}{2}}$ is the Hartmann number. Thus $R_m \ll 1$ provided that $M \ll \epsilon^{-1}$. For mercury in a container of diameter 10^{-1} m this condition will be satisfied provided that

$$B_A \ll 10^3\epsilon^{-1} \text{ weber m}^{-2}.$$

Since the strongest laboratory magnetic fields have intensity of order 1 weber m^{-2} it seems inevitable that $R_m \ll 1$.

The results of § 2 show that an alternating or rotating field will produce a steady vorticity source only in infinitely long cylinders of uniform cross-section, so $\nabla \times \mathbf{F}$ will always in practice be time dependent. An oscillatory term in $\nabla \times \mathbf{F}$ will give rise to an oscillatory flow of order

$$U_1 = \epsilon B_A^2/\rho\mu_0\,\Omega\delta_B$$

(the dominant term in the momentum equation for this flow being $\rho\,\partial\mathbf{u}/\partial t$ for large Ω). The ratio

$$U_1/U = \nu/\lambda = P$$

is the magnetic Prandtl number, which is very small for liquid metals (10^{-7} for mercury). This oscillatory flow is likely to be weak, as pointed out by Moffatt (1965).

The calculation of the boundary-layer flow in case (ii) ($\delta_B \gg \delta_V$) and consideration of the Taylor number seem to indicate that this regime will be unstable. A turbulent flow could develop, or a laminar flow of more complicated geometry with flow in and out of the boundary layer could be established.

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